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Strategic manipulations of multi-valued solutions in economies with indivisibilities

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Abstract

This paper studies strategic manipulations of multi-valued solutions in the problem of fairly allocating homogeneous indivisible objects with monetary transfers. We provide various extensions of strategy-proofness to multi-valued solutions and examine their impact on standard solutions. We show that some efficient and fair solutions, such as the envy-free solution, satisfy certain extensions of strategy-proofness. We also establish an impossibility result on extended strategy-proofness that is defined in terms of expected utility.

Keywords: Extended strategy-proofness; Strategy-proofness; Indivisible goods; Fair allocation; Stochastic choice

JEL classification: C78; D63; D71

1. Introduction

We consider the problem of fairly allocating homogeneous indivisible goods when monetary transfers are possible. A (multi-valued) *solution* is a correspondence which associates with each preference profile a non-empty set of feasible allocations. Our purpose is to study the robustness of multi-valued solutions to strategic manipulations.

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A central property on non-manipulability is *strategy-proofness*, which states that no one can gain by misrepresenting his preferences (Gibbard, 1973; Satterthwaite, 1975). In this context, it is known that no efficient single-valued solution is *strategy-proof* (Green and Laffont, 1977; Holmström, 1979; Schummer, 2000; Ohseto, 2000, 2004; Svensson and Larsson, 2002).[1] However, the definition of *strategy-proofness* is well-defined only for single-valued solutions. Though we do not oppose the desirability of single-valuedness, this property is not weak. In fact, many interesting solutions in this context are defined to be multi-valued. Thus there is room to doubt that the impossibility results come from the underlying assumption of single-valuedness rather than *strategy-proofness* itself. To examine this problem, we consider *extended strategy-proofness* (henceforth, *ESP*) axioms that apply to multi-valued solutions as well as single-valued solutions.

There is no single way to extend *strategy-proofness* to multi-valued solutions. Various ESP axioms have been proposed so far in voting environments.[2] The ESP axioms do not have exactly the same appeal as *strategy-proofness*. For example, they are not always related to dominant strategy implementation unlike *strategy-proofness*. However, every ESP axiom do shares certain important aspects of *strategy-proofness* and coincides with *strategy-proofness* under single-valuedness. Thus checking the satisfaction of ESP axioms for each solution leads to a better understanding of the robustness of the solution to strategic manipulations.

We prove both positive and negative results. They suggest that perfect non-manipulability is impossible but partial non-manipulability is possible under multi-valuedness. One of the main finding of this paper is that the envy-free solution satisfies an ESP axiom that says that, by misrepresentation of preferences, an agent can add an allocation to the initial set of allocations only if the allocation is worse than all originally chosen allocations. Since the envy-free solution is efficient in our model (Svensson, 1983; Alkan et al., 1991), this result contrasts with many impossibility results that show the incompatibility between *strategy-proofness* and some fairness or efficiency notion. On the other hand, the envy-free solution is vulnerable to the manipulation that excludes an undesirable part of the set of chosen allocations. The same possibility and impossibility hold for the efficient and identical preferences lower bound solution.[3] Also, the efficient and egalitarian equivalent solution only satisfies the weakest of our ESP axioms that says that no one can manipulate the solution so that all allocations chosen under manipulation are better than all allocations chosen under the truthful revelation of preferences.[4] Finally, we also prove that no solution mentioned here satisfies an ESP axiom that is based on expected utility on the uniform distribution over a chosen set.

The paper is organized as follows: Section 2 introduces definitions. Section 3 provides ESP axioms. Section 4 presents our main results. Section 5 provides discussions. Finally, Section 6 offers concluding remarks.

[1] Examples of strategy-proof but non-efficient single-valued solutions are Groves solutions (Groves, 1973; Ohseto, 2006) and fixed-price core solutions (Miyagawa, 2001).

[2] See, for example, Pattanaik (1973, 1974), Dutta (1977), Kelly (1977), Gärdenfors (1976, 1979), Barberà (1977a,b), Enelow (1979), and Feldman (1979a,b, 1980). Recent works along this line are Duggan and Schwartz (2000), Barberà et al. (2001), and Ching and Zhou (2002). Klaus and Storcken (2002) study an ESP axiom in the economic problem of choosing the location of a public facility.

[3] This solution associates with each preference profile the set of efficient allocations at which everyone weakly prefers his own bundle to the bundle of the unique envy-free allocation in the hypothetical economy where all agents have the same preferences as him.

[4] This solution associates with each preference profile the set of efficient allocations at which all agents are indifferent between his own bundle to a common reference bundle.

2. Definitions

Let $N \equiv \{1, 2, \ldots, n\}$ be a finite set of *agents*. There are ℓ $(1 \le \ell \le n-1)$ units of homogeneous indivisible good, α.[5] For convenience, we consider that any agent who receives no indivisible good receives a "null" good, β. We allow monetary transfers among the agents. An *allocation* is a pair

$$x \equiv (x_i)_{i \in N} \equiv (\sigma(i), m_i)_{i \in N} \equiv (\sigma, m),$$

where σ: $N \to \{\alpha, \beta\}$ is a function such that $|\sigma^{-1}(\alpha)| = \ell$ and $m \in \mathbb{R}^N$ is a vector such that $\sum_{i \in N} m_i = 0$. Here $\sigma(i)$ is the indivisible object that i receives and $m_i \ge 0$ (resp. $m_i < 0$) is the amount of money he receives (resp. pays). Let X be the set of allocations. Given σ, let $N^{\alpha}(\sigma) \equiv \{i \in N\text{: } \sigma(i) = \alpha\}$ and $N^{\beta}(\sigma) \equiv \{i \in N\text{: } \sigma(i) = \beta\}$.

Each agent $i \in N$ has a valuation over the indivisible good, $v_i \in \mathbb{R}$. Then each i's quasi-linear preference over the consumption space $\{\alpha, \beta\} \times \mathbb{R}$ is represented by, for each $m_i \in \mathbb{R}$,

$$u_i(\alpha, m_i) \equiv v_i + m_i,$$
$$u_i(\beta, m_i) \equiv m_i.$$

A *profile* of valuations is $v \equiv (v_1, v_2, \ldots, v_n) \in \mathbb{R}^N$. Given a profile $v \in \mathbb{R}^N$, let $\overline{v} \in \{v_1, v_2, \ldots, v_n\}$ be the ℓ-th highest number among $v_1, v_2, \ldots, v_n$. Similarly, let $\underline{v}$ be the $(\ell+1)$-th highest number among $v_1, v_2, \ldots, v_n$. For example, when $v_1 \ge v_2 \ge \cdots \ge v_n$, $\overline{v} = v_{\ell}$ and $\underline{v} = v_{\ell+1}$.

A *solution* is a correspondence from $\mathbb{R}^N$ to X which associates with each profile $v \in \mathbb{R}^N$ a non-empty set of allocations, $\psi(v) \subseteq X$. Given two solutions ψ and ϕ, we write $\psi\phi$ to denote the intersection of ψ and ϕ. That is, for each $v \in \mathbb{R}^N$, $\psi\phi\,(v) \equiv \psi(v) \cap \phi(v)$. Also, we write $\psi \subseteq \phi$ if, for each $v \in \mathbb{R}^N$, $\psi(v) \subseteq \phi(v)$, and write $\psi \subsetneq \phi$ if $\psi \subseteq \phi$ and for some $v \in \mathbb{R}^N$, $\psi(v) \subsetneq \phi(v)$. Given $i \in N$, $\psi_i(v) \equiv \{x_i \in \{\alpha, \beta\} \times \mathbb{R} : x \in \psi(v)\}$ is the restriction of $\psi(v)$ on i.

We first define a familiar efficiency solution:

The (Pareto-)efficient solution, P: An allocation $x \in X$ is *efficient* for $v \in \mathbb{R}^N$ if there exists no $y \in X$ such that for each $i \in N$, $u_i(y_i) \ge u_i(x_i)$ and for some $j \in N$, $u_j(y_j) > u_j(x_j)$. Let $P(v)$ be the set of efficient allocations for v.

By quasi-linearity of preferences, an allocation (σ, m) is efficient for v if and only if

$$\min_{i \in N^{\alpha}(\sigma)} v_i \ge \max_{i \in N^{\beta}(\sigma)} v_i.$$

3. Extended strategy-proofness

3.1. Preliminary

We first define a central non-manipulability axiom that is well-defined only for single-valued solutions. It states that no one can gain by preference misrepresentation:

Strategy-proofness: A solution ψ satisfies *strategy-proofness* if there exist no $v \in \mathbb{R}^N$, $i \in N$ and $v_i' \in \mathbb{R}$ such that

$$u_i(\psi_i(v_i', v_{-i})) > u_i(\psi_i(v)).$$

[5] This economy is studied by Tadenuma and Thomson (1991). A special case with only one object (i.e., $\ell = 1$) is intensively analyzed by Tadenuma and Thomson (1993, 1995). We refer to Thomson (2004) for a survey.

We shall provide extensions of *strategy-proofness* so as to be compatible with multi-valuedness. Hereafter we restrict our attention to compact-valued solutions, that is, solutions such that $\psi(v)$ is compact for each $v \in \mathbb{R}^N$. When we say that a solution ψ satisfies an ESP axiom, the solution is supposed to be compact-valued. This is just to avoid some technical problems such as the lack of maximum or minimum and to simplify definitions of ESP axioms.[6] All standard "fair" solutions studied in this paper and in the literature are compact-valued.[7]

To define ESP axioms, it is often useful to consider preferences over sets of allocations. Since we restrict our attention to compact-valued solutions, it suffices to extend preferences to compact sets of allocations. Let

$$\mathcal{X} \equiv \{A \in 2^X \setminus \{\emptyset\} : A \text{ is compact}\}$$

be the set of all non-empty compact sets of allocations. For each $i \in N$, we denote by $\succ_{u_i}$ a strict preference on $\mathcal{X}$ that is defined by u_i. We do not deal with its indifference part, since it is not necessary in order to define ESP axioms.

3.2. Minimum and maximum

The following axiom is studied by Nehring (2000) under the name of "generalized strategy-proofness".[8] As he says, this axiom is at the edge of possibility: if it is weakened, the *strategy-proofness* interpretation is lost.

Separation-ESP: Given u_i, let $\succ^s_{u_i}$ be such that, for each $A, B \in \mathcal{X}$,

$$A \succ^s_{u_i} B \text{ iff } \min_{a \in A} u_i(a_i) > \max_{b \in B} u_i(b_i).$$

A solution ψ satisfies *separation-ESP* if there exist no $v \in \mathbb{R}^N$, $i \in N$, and $v_i' \in \mathbb{R}$ such that

$$\psi_i(v_i', v_{-1}) \succ^s_{u_i} \psi_i(v).$$

The next axiom is based on the maximin criterion.[9]

Min-ESP: Given u_i, let $\succ^{\min}_{u_i}$ be such that, for each $A, B \in \mathcal{X}$,

$$A \succ^{\min}_{u_i} B \text{ iff } \min_{a \in A} u_i(a_i) > \min_{b \in B} u_i(b_i).$$

A solution ψ satisfies *min-ESP* if there exist no $v \in \mathbb{R}^N$, $i \in N$ and $v_i' \in \mathbb{R}$ such that

$$\psi_i(v_i', v_{-i}) \succ^{\min}_{u_i} \psi_i(v).$$

[6] For example, under the compactness assumption, the maximum and minimum of utilities over the set of consumption bundles exist. Also, the uniform distribution on any compact set of allocations is well-defined. This also enables us to state notation and definitions in a much simpler way. For example, when ψ is compact-valued, we can write "$\max_{x \in \psi(v)} u_i(x) < \max_{y \in \psi(v')} u_i(y)$" instead of "for each $x \in \psi(v)$, there exists $y \in \psi(v')$ such that $u_i(x_i) < u_i(y_i)$". This facilitates the understanding of the discussion since the definitions of ESP axioms are relatively complicated.

[7] However, one can handle non compact-valued solutions in a similar way.

[8] He shows that this condition is necessary for solutions to be implemented in Nash equilibrium in voting environments.

[9] Strategic manipulation based on the maximin criterion is analyzed by Pattanaik (1973), Dutta (1977), and Thomson (1979).

The next condition, which is based on the maximax criterion, is introduced by Jackson (1992).[10]

Max-ESP: Given u_i, let $\succ_{u_i}^{\max}$ be such that, for each $A, B \in \mathcal{X}$,

$$A \succ_{u_i}^{\max} B \text{ iff } \max_{a \in A} u_i(a_i) > \max_{b \in B} u_i(b_i).$$

A solution ψ satisfies *min-ESP* if there exist no $v \in \mathbb{R}^N$, $i \in N$, and $v_i' \in \mathbb{R}$ such that

$$\psi_i(v_i', v_{-i}) \succ_{u_i}^{\max} \psi_i(v).$$

3.3. No addition and deletion

Ching and Zhou (2002) introduce two ESP axioms, each of which implies *max-ESP* and *min-ESP*, respectively. They call the pair of the two axioms "strategy-proofness". The first one states that if an allocation that was not originally chosen is chosen following an agent's misrepresentation, then the allocation is worse than all originally chosen allocations for the agent. The second one states that if an allocation that was originally chosen is not chosen following an agent's misrepresentation, then the allocation is better than all allocations chosen under the manipulation.

No-addition-ESP: There exist no $v \in \mathbb{R}^N$, $i \in N$, and $v_i' \in \mathbb{R}$ such that for some $y \in \psi(v_i', v_{-i}) \setminus \psi(v)$,

$$u_i(y_i) > \min_{x \in \psi(v)} u_i(x_i).$$

Any ψ satisfying *no-addition-ESP* is robust to the strategic enlargement of the set of chosen allocations. We say that an agent i can *enlargement-manipulate* ψ at $v \in \mathbb{R}^N$ if there exists $v_i' \in \mathbb{R}$ such that $\psi(v) \subsetneq \psi(v_i', v_{-i})$ and for each $x \in \psi(v_i', v_{-i}) \setminus \psi(v)$ and each $y \in \psi(v)$, $u_i(x_i) > u_i(y_i)$. If ψ satisfies *no-addition-ESP*, then there are no $i \in N$ and $v \in \mathbb{R}^N$ such that i can *enlargement-manipulate* ψ at v.

No-deletion-ESP: There exist no $v \in \mathbb{R}^N$, $I \in N$, and $v_i' \in \mathbb{R}^N$ such that there exists $x \in \psi(v) \setminus \psi(v_i', v_{-i})$ for which

$$\max_{y \in \psi(v_i', v_{-i})} u_i(y_i) > u_i(x_i).$$

Any ψ satisfying *no-deletion-ESP* is robust to the strategic reduction of the set of chosen allocations. We say that an agent $i \in N$ can *reduction-manipulate* ψ at $v \in \mathbb{R}^N$ if there exists $v_i' \in \mathbb{R}$ such that $\psi(v_i', v_{-i}) \subsetneq \psi(v)$ and for each $x \in \psi(v) \setminus \psi(v_i', v_{-i})$ and each $y \in \psi(v_i', v_{-i})$, $u_i(y_i) > u_i(x_i)$. If ψ satisfies *no-deletion-ESP*, then there are no $i \in N$ and $i \in \mathbb{R}^N$ such that i can *reduction-manipulate* ψ at v.

3.4. Expected utility

Given $v \in \mathbb{R}^N$, since $\psi(v)$ is the set of best allocations in views of ψ, whenever ψ is applied, it is natural to consider that all allocations in $\psi(v)$ are equally desirable. Then it makes sense to use the

[10] He derives this condition as a necessary condition for multi-valued solutions to be implemented in undominated strategies using bounded mechanisms.

uniform distribution over $\psi(v)$ to select one final allocation from this set. This idea justifies the following ESP axiom for the case when agents evaluate sets of allocations by expected utility:[11]

Expected-utility-ESP: Given u_i and $A \in \mathcal{X}$, let $E(u_i : A)$ be the expected utility of u_i at A with respect to the uniform distribution f over A, i.e.),

$$E(u_i : A) \equiv \int_{x \in A} u_i(x_i) f(x) \mathrm{d}x.$$

A solution ψ satisfies *expected-utility-ESP* if there exist no $v \in \mathbb{R}^N$, $i \in N$, and $v_i' \in \mathbb{R}$ such that

$$E(u_i : \psi(v_i', v_{-i})) > E(u_i : \psi(v)).$$

Since *expected-utility-ESP* only depends on the average of utilities over allocations, it implies none of the ESP axioms defined so far except for *separation-ESP*. *Expected-utility-ESP* is introduced by Feldman (1979a, 1980) in voting environments.[12] Feldman (1979a, 1980) and Barberà, Dutta, and Sen (2001) study this axiom and obtain certain dictatorship results in voting environments. Gärdenfors (1979) analyzes relations between preferences on sets and expected utility functions when distributions over the set of chosen allocations are unknown.

Lemma 1. *The following relations hold for ESP axioms:*

(i) No-addition-ESP ⇒ max-ESP ⇒ separation-ESP;
(ii) No-deletion-ESP ⇒ min-ESP ⇒ separation-ESP;
(iii) Expected-utility-ESP ⇒ separation-ESP.

4. Manipulability of fair solutions

4.1. Envy-freeness

A central fairness concept in the theory of fair allocation is envy-freeness (Foley, 1967). It states that everyone should weakly prefer his own consumption bundle to anyone else's at any chosen allocation.

The envy-free solution, F: An allocation $x \in X$ is *envy-free* for $v \in \mathbb{R}^N$ if for each $i, j \in N$, $u_i(x_i) \geq u_i(x_j)$. Let $F(v)$ be the set of envy-free allocations for v.

It is known that for each $v \in \mathbb{R}^N$, $F(v) \neq \emptyset$ (Alkan et al., 1991, Theorem 2).[13] In this context, all envy-free allocations are efficient (Svensson, 1983, Theorem 2). This relation much strengthens the appeal of the envy-free solution. The next lemma is useful to understand the structure of this solution:

Lemma 2. *For each $v \in \mathbb{R}^N$, $(\sigma, m) \in F(v)$ if and only if*

(i) for each $i, j \in N^{\alpha}(\sigma)$, $m_i = m_j$,
(ii) for each $i, j \in N^{\beta}(\sigma)$, $m_i = m_j$,
(iii) when $k \in \arg\min_{i \in N^{\alpha}(\sigma)} v_i$ and $h \in \arg\max_{i \in N^{\beta}(\sigma)} v_i$, we have $v_k \geq v_h$,

$$m_k \in \left[-\frac{n-\ell}{n} v_k, -\frac{n-\ell}{n} v_h \right] \text{ and } m_h \in \left[\frac{\ell}{n} v_h, \frac{\ell}{n} v_k \right].$$

[11] Note that the uniform distribution on $\psi(v)$ is well-defined since this set is compact.
[12] Definition 7 in Feldman (1979a) and "non-manipulability" in Feldman (1980) correspond to this.
[13] In our simpler setting, the existence of envy-free allocations can be derived as a corollary to our Lemma 2.

Proof. Let $x \equiv (\sigma, m) \in F(v)$. (i) and (ii) are obvious. Let us verify (iii). Since $x \in P(v)$, when $k \in \arg\min_{i \in N^\alpha(\sigma)} v_i$ and $h \in \arg\max_{i \in N^\beta(\sigma)} v_i$, we have $v_k \geq v_h$. Note that $u_k\left(\alpha, -\frac{n-\ell}{n} v_k\right) = u_k\left(\beta, \frac{\ell}{n} v_k\right)$ and $u_h\left(\alpha, -\frac{n-\ell}{n} v_h\right) = u_h\left(\beta, \frac{\ell}{n} v_h\right)$. If $-\frac{n-\ell}{n} v_k > m_k$, then agent k envies agent h. If $m_k > -\frac{n-\ell}{n} v_h$, then agent h envies agent k. Therefore, $m_k \in \left[-\frac{n-\ell}{n} v_k, -\frac{n-\ell}{n} v_h\right]$, and by feasibility, $m_h \in \left[\frac{\ell}{n} v_h, \frac{\ell}{n} v_k\right]$.

Conversely, let $(\sigma, m) \in X$ satisfy (i)–(iii). Let $i, j \in N$. If either $i, j \in N^\alpha(\sigma)$ or $i, j \in N^\beta(\sigma)$, then by (i) and (ii), $u_i(x_i) = u_i(x_j)$. Hence, suppose that $i \in N^\alpha(\sigma)$ and $j \in N^\beta(\sigma)$. When $k \in \arg\min_{i \in N^\alpha(\sigma)} v_i$ and $h \in \arg\max_{i \in N^\beta(\sigma)} v_i$, by $v_i \geq v_k$ and (iii), $u_i(\alpha, m_i) \geq u_i\left(\alpha, -\frac{n-\ell}{n} v_k\right) \geq u_i\left(\beta, \frac{\ell}{n} v_k\right) \geq u_i(\beta, m_j)$. Similarly, by $v_h \geq v_j$ and (iii), $u_j(\beta, m_j) \geq u_j\left(\beta, \frac{\ell}{n} v_h\right) \geq u_j\left(\alpha, -\frac{n-\ell}{n} v_h\right) \geq u_j(\alpha, m_i)$. Therefore, $(\sigma, m) \in F(v)$. □

This lemma implies that the best envy-free allocation for i with $v_i \geq \overline{v}$ (resp. $\underline{v} \geq v_i$) is determined by $\underline{v}$ (resp. $\overline{v}$). This fact suggests that no agent who receives α can manipulate the envy-free solution in such a way that his best envy-free allocation is improved. Also, it suggests that enlarging the set of envy-free allocations in a favored way is impossible.

Proposition 1. *F satisfies no-addition-ESP.*

Proof. Let $v \in \mathbb{R}^N$. We only consider the simple case that v has no tie: for each $i, j \in N$, $v_i \neq v_j$. The case with tie can be handled in the same way as non-tie cases, but only dealing with the non-tie case suffices to understand the essence of the proof.

Without loss of generality, assume that $v_1 < v_2 < \cdots < v_n$. Let $j \in N$. We assume that $v_j \geq v_{n-\ell+1}$ (i.e., $v_j \geq \overline{v}$). The other case can be handled in a parallel way. Let $v_j' \neq v_j$. Note that

$$F_j(v) = \left\{(\alpha, m_j) : m_j \in \left[-\frac{n-\ell}{n} v_{n-\ell+1}, -\frac{n-\ell}{n} v_{n-\ell}\right]\right\}$$

and $\min_{x \in F(v)} u_j(x_j) \geq \frac{\ell}{n} v_{n-\ell+1}$.

Case (i) $v_j > v_{n-\ell+1}$ (i.e., $j > n-\ell+1$):
Subcase (i-i) $v_{n-\ell+1} \leq v_j'$: Then, $F_j(v) = F_j(v_j', v_{-j})$.
Subcase (i-ii): $v_{n-\ell} < v_j' < v_{n-\ell+1}$: In this subcase, $F_j(v_j', v_{-j}) \subsetneq F_j(v)$.
Subcase (i-iii) $v_{n-\ell} = v_j' < v_{n-\ell+1}$: Then,

$$F_j(v_j', v_{-j}) = \left\{\left(\alpha, -\frac{n-\ell}{n} v_{n-\ell}\right), \left(\beta, \frac{\ell}{n} v_{n-\ell}\right)\right\}.$$

Hence, $F_j(v_j', v_{-j}) \setminus F_j(v) = \left\{\left(\beta, \frac{\ell}{n} v_{n-\ell}\right)\right\}$. Since

$$u_j\left(\beta, \frac{\ell}{n} v_{n-\ell}\right) = \frac{\ell}{n} v_{n-\ell} < \frac{\ell}{n} v_{n-\ell+1} \leq \min_{x_j \in F_j(v)} u_j(x_j),$$

we obtain the desired result.

Subcase (i-iv) $v_j' < v_{n-\ell} < v_{n-\ell+1}$: In this case, for each $(\sigma, m) \in F(v_j', v_{-j})$,

$$\sigma(j) = \beta \text{ and } m_j \leq \frac{\ell}{n} v_{n-\ell}.$$

Hence,

$$\max_{y \in F(v_j', v_j)} u_j(y_j) \leq \frac{\ell}{n} v_{n-\ell} \leq \min_{x \in F(v)} u_j(x_j).$$

Case (ii) $v_j' = v_{n-\ell+1}$ (i.e., $j = n-\ell+1$):
Subcase (ii-i) $v_j < v_j' \leq v_{j+1}$: Then,

$$F_j(v_j', v_{-j}) = \left\{ (\alpha, m_j) : m_j \in \left[-\frac{n-\ell}{n} v_j', -\frac{n-\ell}{n} v_{n-\ell} \right] \right\}.$$

Hence, for each $y \in F(v_j', v_{-j}) \backslash F(v)$,

$$u_j(y_j) \leq \min_{x \in F(v)} u_j(x_j).$$

Subcase (ii-ii) $v_{n-\ell} < v_j' < v_j$: In this case, $F_j(v_j', v_{-j}) \subsetneq F_j(v)$.
Subcase (ii-iii) $v_{n-\ell} = v_j'$: The same argument as (i-iii) holds, so we omit the proof.
Subcase (ii-iv) $v_j' < v_{n-\ell}$: The same argument as (i-iv) holds, so we omit the proof. □

Proposition 1 contrasts with many impossibility results in this literature. In particular, it is known that, when there are two agents, the class of single-valued and *strategy-proof* solutions can be characterized by certain forms of constancy or dictatorship (Schummer, 2000, Theorem 1).[14] Also, when there is only one real object, even on finitely restricted domains, there is no single-valued subsolution of the efficient solution satisfying *strategy-proofness* and *equal compensation* (Ohseto, 2000, Theorem 1).[15] The envy-free solution is a subsolution of the efficient solution that satisfies *equal compensation* and do not exhibit any constancy or dictatorship. Hence, our theorem implies that we can somewhat escape Schummer and Ohseto's impossibility results if we allow solutions to be multi-valued and extend *strategy-proofness* to *no-addition-ESP*. However, the envy-free solution escapes some but not all form of manipulation: the next proposition is negative and shows that the envy-free solution is almost always vulnerable to *reduction–manipulation*.

Proposition 2. *For each $v \in \mathbb{R}^N$, F is reduction–manipulable at v if and only if $\overline{v} > \underline{v}$.*

Proof. Let $v \in \mathbb{R}^N$. If $\overline{v} > \underline{v}$, then by Lemma 2, any $i \in N$ can *reduction-manipulate* F by reporting v_i' with $\overline{v} > v_i' > \underline{v}$. Also, if $\overline{v} = \underline{v}$, then by Lemma 2, $F(v)$ is essentially singleton and no $i \in N$ can *reduction-manipulate* here. □

4.2. Identical preferences lower bound

The notion of the identical preferences lower bound states that everyone should benefit from the diversity of preferences (Moulin, 1990). Given $v_i \in \mathbb{R}$, let $r(v_i) \in X$ be such that for each $j \in N$, $u_i(r_i(v_i)) = u_i(r_j(v_i))$. This is an allocation that treats everyone symmetrically when all agents have the same preference as i. Note that such an $r(v_i)$ is essentially unique in that if $r(v_i)$ and $r'(v_i)$ are two such reference allocations, $u_i(r_i(v_i)) = u_i(r_i'(v_i))$. Slightly abusing notation, we deal with $r(v_i)$ as if it were a consumption bundle.

Identical preferences lower bound solution, B: For each $v \in \mathbb{R}^N$, let $B(v) \equiv \{x \in X$: For each $i \in N$, $u_i(x_i) \geq u_i(r(v_i))\}$.

It is known that if $n=2$, then $F=B$, and if $n>2$, then $F \subsetneq B$ (Moulin, 1990, p152; Beviá, 1996, Propositions 1 and 2). The following is a characterization of reference bundles for the identical preferences' lower bound.

[14] His theorem deals with the two-agents and two-objects case, while we deal with the case where the number of objects is smaller than the number of agents. However, in the two-agent case, these two cases are equivalent, since our α and β can be regarded as two different indivisible objects.

[15] *Equal compensation*: For each $v \in \mathbb{R}^N$ and each $(\sigma, m) \in \psi(v)$, if $i, j \in N$ are such that $\sigma(i) = \sigma(j) = \beta$, then $m_i = m_j$. Ohseto (1999) also studies this axiom together with *strategy-proofness*.

Lemma 3. *For each* $v_i \in \mathbb{R}^N$, $r(v_i) = (\alpha, -\frac{n-\ell}{n} v_i)$ *(or, alternately,* $(\beta, \frac{\ell}{n} v_i)$*).*

Proof. Since the set of symmetric allocations for $(v_i, v_i, \ldots, v_i)$ coincides with the set of envy-free allocations $F(v_i, v_i, \ldots, v_i)$, this immediately follows from Lemma 2. □

Lemma 3 implies the following characterizations of $PB_i(v)$:

Lemma 4. *For each* $v \in \mathbb{R}^N$, *let* σ *be such that*[16]

$$\min_{i \in N^{\alpha}(\sigma)} v_i \geq \max_{i \in N^{\beta}(\sigma)} v_i.$$

Then, for each $i \in N$,

(i) if $v_i > \overline{v}$ or $v_i = \overline{v} > \underline{v}$, then

$$PB_i(v) = \left\{ (\alpha, m_i) : m_i \in \left[-\frac{n-\ell}{n} v_i, \sum_{j \in N^{\alpha}(\sigma) \setminus \{i\}} v_j - \frac{\ell}{n} \left(\sum_{j \in N^{\beta}(\sigma)} v_j + \sum_{j \in N^{\alpha}(\sigma) \setminus \{i\}} v_j \right) \right] \right\},$$

(ii) if $v_i < \underline{v}$ or $v_i = \underline{v} < \overline{v}$, then

$$PB_i(v) = \left\{ (\beta, m_i) : m_i \in \left[\frac{\ell}{n} v_i, \sum_{j \in N^{\alpha}(\sigma)} v_j - \frac{\ell}{n} \left(\sum_{j \in N^{\beta}(\sigma) \setminus \{i\}} v_j + \sum_{j \in N^{\alpha}(\sigma)} v_j \right) \right] \right\},$$

(iii) if $v_i = \underline{v} = \overline{v}$, then $PB_i(v)$ is the union of the two sets in (i) and (ii).

We provide a slightly weaker version of *no-addition-ESP*. It states that, at every true preference profile, when an agent manipulates a solution, the welfare level he achieves is lower than all achievable welfare levels under the true preference profile.

Weak no-addition-ESP: There are no $v \in \mathbb{R}^N$, $i \in N$, and $v_i' \in \mathbb{R}$ such that for some $y \in \psi(v_i', v_{-i}) \setminus \psi(v)$ with $u_i(y_i) \notin \{u_i(z_i) \in \mathbb{R} : z \in \psi(v)\}$, $u_i(y_i) > \min_{x \in \psi(v)} u_i(x_i)$.

Note that in the definition of *no-addition-ESP*, the term "$u_i(y_i) \notin \{u_i(z_i) \in \mathbb{R} : z \in \psi(v)\}$" does not appear. This is the only difference between *no-addition-ESP* and *weak no-addition-ESP*. However, since agents are only concerned with welfare levels, we consider that this relaxation does not lose the essence of *no-addition-ESP*.

Proposition 3. *If* $n = 2$, *then PB satisfies no-addition-ESP. If* $n \geq 3$, *then PB does not satisfy no-addition-ESP, but it satisfies weak no-addition-ESP.*

Proof. If $n = 2$, PB coincides with F (Beviá, 1996, Proposition 1), hence by Proposition 1, F satisfies *no-addition-ESP*.

Assume $n \geq 3$. Since $l < n$, at least one of $l \geq 2$ or $n - \ell \geq 2$ holds. We only prove the case $\ell \geq 2$. The other case can be proved in a parallel way.

Let us show the first part of the claim. Let

$$v \equiv \left(\frac{\ell}{2n}, \underbrace{1, 1, \ldots, 1}_{\ell-1}, \underbrace{0, 0, \ldots, 0}_{n-\ell} \right)$$

[16] There may be multiple such σ. However, the result is independent of the choice of σ.

be the true profile of valuations. Let $v'_1 \equiv 0$. Agent 1's worst allocation among $PB(v)$ is $x \in PB(v)$ with

$$x_1 \equiv \left(\alpha, -\frac{n-\ell}{n}\frac{\ell}{2n}\right).$$

There exists $y \equiv (\sigma, m) \in PB(v'_1, v_{-1})$ with

$$y_1 = (\sigma(1), m_1) = \left(\beta, \frac{n-\ell}{n}\sum_{N^\alpha(\sigma)} v_j - \frac{\ell}{n}\sum_{j \in N^\beta(\sigma)\setminus\{1\}} v_j\right) = \left(\beta, \frac{n-\ell}{n}(\ell-1)\right).$$

Since $y \notin PB(v)$, it suffices to show that $u_1(x_1) < u_1(y_1)$.

One can easily see that

$$u_1(x_1) < u_1(y_1) \Leftrightarrow \frac{\ell}{n}\frac{\ell}{2n} < \frac{n-\ell}{n}(\ell-1) \Leftrightarrow \frac{1}{2}\frac{\ell}{n} < \frac{\ell-1}{\ell}(n-\ell). \tag{1}$$

Since $\ell \geq 2$, $\frac{1}{2} \leq \frac{\ell-1}{\ell}$. This and $\frac{\ell}{n} < n-\ell$ together imply the third inequality of Eq. (1). Thus $u_1(x_1) < u_1(y_1)$ holds.

For the second part of the claim, we only show that in the above counterexample, there exists $z_1 \in PB_1(v)$ such that $u_1(z_1) = u_1(y_1)$, so the existence of y_1 does not lead to the violation of *weak no-addition-ESP*. A formal proof can be easily obtained by applying the same argument, so we omit it. Agent 1's best allocation among the set $PB_1(v)$ is $w \in PB_1(v)$ with $w_1 = \left(\alpha, (\ell-1)\frac{n-\ell}{n}\right)$. By a routine calculation, $u_1(y_1) < u_1(w_1)$. Thus, $u_1(x_1) < u_1(y_1) < u_1(w_1)$. Hence, there exists $m'_1 \in \left(-\frac{n-\ell}{n}\frac{\ell}{2n}, (\ell-1)\frac{n-\ell}{n}\right)$ such that $u_1(y_1) = u_1(\alpha, m'_1)$. Since $z_1 \equiv (\alpha, m'_1) \in PB_1(v)$, this completes the proof. □

It is easy to see that, if ψ satisfies *weak no-addition-ESP*, then it is *enlargement-non-manipulable* at all $v \in \mathbb{R}^N$. Thus Proposition 3 implies that PB satisfies this non-manipulability condition. On the other hand, for *reduction–manipulation*, the same impossibility as F holds:

Proposition 4. *For each $v \in \mathbb{R}^N$, PB is reduction-manipulable at v if and only if $\overline{v} > \underline{v}$.*

Proof. This can be easily shown using Lemma 4. □

We next provide a strong impossibility result. It establishes the non-existence of $\psi \subseteq PB$ satisfying *expected-utility-ESP*.

Proposition 5. *There exists no solution $\psi \subseteq PB$ satisfying expected-utility-ESP.*

Proof. Suppose, by contradiction, that there exists $\psi \subseteq PB$ satisfying *expected-utility-ESP*. Let

$$v \equiv \left(\underbrace{1, 1, \ldots, 1}_{\ell}, \underbrace{0, 0, \ldots, 0}_{n-\ell}\right) \in \mathbb{R}^N.$$

Claim 1. **For each $i \leq \ell$, $E(u_i : \psi(v)) \geq 1$.** Let $i \leq \ell$. Note $v_i = 1$. It suffices to show that for arbitrary small $\varepsilon > 0$,

$$E(u_i : \psi(v)) \geq 1 - \varepsilon. \tag{2}$$

Let $\varepsilon > 0$ and $v'_i \equiv \varepsilon$.

By *expected-utility-ESP*,

$$E(u_i : \psi(v)) \geq E(u_i : \psi(v_i', v_{-i})). \tag{3}$$

By definition,

$$E(u_i : \psi(v_i', v_{-i})) \geq \min_{x' \in \psi(v_i', v_{-i})} u_i(x_i'). \tag{4}$$

We claim that

$$\min_{x' \in \psi(v_i', v_{-i})} u_i(x_i') \geq 1 - \varepsilon. \tag{5}$$

Since $\psi(v_i', v_{-i}) \subseteq PB(v_i', v_{-i})$, for each $x' = (\sigma', m') \in \psi(v_i', v_{-i})$,

$$v_i' + m_i' = u_i'(x_i') \geq \frac{\ell}{n} v_i', \tag{6}$$

$$m_i' \geq -\frac{n-\ell}{n} v_i', \tag{7}$$

$$\text{so } u_i(x_i') = 1 + m_i' \geq 1 - \frac{n-\ell}{n} v_i' \geq 1 - \varepsilon. \tag{8}$$

Since Eq. (8) holds for arbitrary chosen $x' \in \psi(v_i', v_{-i})$, we have Eq. (5). Thus, by Eqs. (3), (4), and (5), we have Eq. (2). Since Eq. (2) was proven for arbitrary small $\varepsilon > 0$, the claim is now established.

Claim 2. Some $i \geq \ell + 1$ can manipulate. Note that, when f is the uniform distribution on $\psi(v)$,

$$\sum_{i \in N} E(u_i : \psi(v)) = \sum_{i \in N} \int_{x \in \psi(v)} u_i(x_i) f(x) \mathrm{d}x = \int_{x \in \psi(v)} f(x) \Big(\sum_{x \in N} u_i(x_i)\Big) \mathrm{d}x$$
$$= \int_{x \in \psi(v)} f(x) \ell \mathrm{d}x = \ell \int_{x \in \psi(v)} f(x) \mathrm{d}x = \ell.$$

Therefore, by Claim 1, there exists $i \geq \ell + 1$ such that $E(u_i : \psi(v)) \leq 0$. We claim that i can manipulate ψ at v. For $v_i' \in (0, 1)$, since $\psi(v_i', v_{-i}) \in PB(v_i', v_{-i})$,

$$E(u_i : \psi(v)) \leq 0 < \frac{\ell}{n} v_i' \leq \min_{(\sigma', m') \in PB(v_i', v_{-i})} m_i' \leq \min_{(\sigma', m') \in \psi(v_i', v_{-i})} m_i'$$
$$= \min_{(\sigma', m') \in \psi(v_i', v_{-i})} u_i(\beta, m_i') \leq E(u_i : \psi(v_i', v_{-i})).$$

This contradicts *expected-utility-ESP*. □

Since $F \subseteq PB$, Proposition 5 implies that no $\psi \subseteq F$ satisfies *expected-utility-ESP*.

4.3. Egalitarian equivalence

The notion of egalitarian equivalence (Pazner and Schmeidler, 1978) states that each agent should receive a consumption bundle that is indifferent to a common "reference" consumption bundle. Given $v \in \mathbb{R}^N$, an allocation x is *α-egalitarian equivalent* for v if there exists $m_\alpha^* \in \mathbb{R}$ such that for each $i \in N$, $u_i(x_i) = u_i(\alpha, m_\alpha^*)$. Let $E^\alpha(v)$ be the set of α-egalitarian equivalent allocations for v. Similarly, x is *β-egalitarian equivalent* for v if there exists $m^*_\beta \in \mathbb{R}$ such that for each $i \in N$, $u_i(x_i) = u_i(\alpha, m^*_\alpha)$. Let $E^\alpha(v)$ be the set of β-egalitarian equivalent allocations for v.

Egalitarian-equivalent solution, E: For each $v \in \mathbb{R}^N$, let $E(v) \equiv E^\alpha(v) \cup E^\beta(v)$ be the set of egalitarian equivalent allocations.

The following is a characterization of the set of α-egalitarian-equivalent allocations:

Lemma 5. *For each* $v \in \mathbb{R}^N$, *(σ, m)* $\in E^\alpha(v)$ *if and only if*

$$m_i = -\frac{\sum_{j \in N^\beta(\sigma)} v_j}{n} \text{ for each } i \in N^\alpha(\sigma),$$

$$= v_i - \frac{\sum_{j \in N^\beta(\sigma)} v_j}{n} \text{ for each } i \in N^\beta(\sigma).$$

Proof. We only prove the *only if* part, since the other part is straightforward. Let $(\sigma, m) \in E^\alpha(v)$. Without loss of generality, assume that $N^\beta(\sigma) = \{1, 2, \ldots, n - \ell\}$. Since $(\sigma, m) \in E^\alpha(v)$, all agents in $N^\alpha(\sigma)$ receive the same amount of money, $c \in \mathbb{R}$. For each $i \in N^\beta(\sigma)$, since $u_i(\alpha, c) = u_i(\beta, m_i)$, we have that $m_i - v_i = c$. Note that for each $i, j \in N^\beta(\sigma)$ with $v_i \leq v_j$, m_j is greater than m_i exactly by $v_j - v_i$. Therefore,

$$\sum_{i=1}^{n-\ell} m_i = \sum_{i=1}^{n-\ell} v_i + (n-\ell)c.$$

By budget balancedness, $(n-\ell)c + \sum_{i=1}^{n-\ell} v_i + \ell c = 0$. Hence,

$$c = -\frac{\sum_{i=1}^{n-\ell} v_i}{n}. \quad \square$$

This lemma implies that any α-egalitarian equivalent allocation is characterized by valuations of agents who receive β. Thus, whenever the set $N^\beta(\sigma)$ is unchanged, any agent $i \in N^\beta(\sigma)$ can increase his money by reporting $v_i' > v_i$. This fact underlines the main difficulty with egalitarian-equivalent solutions concerning the satisfaction of ESP axioms.

Since α-egalitarian equivalence and β-egalitarian equivalence are symmetric, the characterization of β-egalitarian equivalent allocations is also obtained in a similar fashion:

Lemma 6. *For each* $v \in \mathbb{R}^N$, $(\sigma, m) \in E^\beta(v)$ *if and only if*

$$m_i = -v_i + \frac{\sum_{j \in N^\alpha(\sigma)} v_j}{n} \text{ for each } i \in N^\alpha(\sigma),$$

$$= \frac{\sum_{j \in N^\alpha(\sigma)} v_j}{n} \text{ for each } i \in N^\beta(\sigma).$$

Proposition 6. *PE^α and PE^β do not satisfy separation-ESP; PE satisfies separation-ESP, but not min-ESP, max-ESP, and expected-utility-ESP.*

Proof. We only show that *PE* satisfies *separation-ESP*, since other proofs are easy. Let $v \in \mathbb{R}^N$, $i \in N$, and $v_i' \neq v_i$. We assume that $v_i \geq \underline{v}$. The other case can be proven in a parallel way. Note that

$$x_i = \left(\alpha, -\frac{\sum_{j \in N^\beta(\sigma')} v_j}{n} \right) \in PE_i(v).$$

Let $x \equiv (\sigma, m)$ be the efficient and α-egalitarian equivalent allocation for v with $\sigma(i) = \alpha$. Note that $x \in PE(v)$. We shall show that there is $y \in PE(v_i', v_{-i})$ such that $u_i(x_i) \geq u_i(y_i)$.

If $v_i' \geq \underline{v}$, then $x \in PE(v_i', v_{-i})$. Hence, let $y \equiv x$ in this case. Suppose that $v_i' < \underline{v}$. Let $v' \equiv (v_i', v_{-i})$. Let $y \equiv (\sigma', m')$ be the efficient and α-egalitarian equivalent allocation for v' with

$$y_i = \left(\beta, v_i' - \frac{\sum_{j \in N^\beta(\sigma')} v_j'}{n} \right).$$

Note that $y \in PE(v')$.

One can easily see that

$$\begin{aligned} u_i(x_i) > u_i(y_i) &\Leftrightarrow v_i - \frac{\sum_{j \in N^\beta(\sigma)} v_j}{n} > v_i' - \frac{\sum_{j \in N^\beta(\sigma')} v_j'}{n} \\ &\Leftrightarrow v_i - v_i' > \frac{\sum_{j \in N^\beta(\sigma)} v_j}{n} - \frac{\sum_{j \in N^\beta(\sigma')} v_j'}{n} \\ &\Leftrightarrow v_i - v_i' > \frac{\underline{v} - v_i'}{n}. \end{aligned}$$

Since $v_i \geq \underline{v} > v_i'$, we have

$$v_i - v_i' > \frac{\underline{v} - v_i'}{n}.$$

Hence, $u_i(x_i) > u_i(y_i)$. □

The only positive result we obtained is that *PE* satisfies *separation-ESP*, which is the weakest extension of *strategy-proofness*. Thus Proposition 6 is rather negative.

5. Discussion

5.1. Other ESP axioms

There are many other ESP axioms in the literature. We refer to Gärdenfors (1979), Feldman (1979a,b), and Klaus and Storcken (2002) for the interested reader. We here only mention an ESP axiom whose flavor is quite different from ours. The axiom is studied by Tadenuma and Thomson (1995) in economies with indivisibilities. It states that there exists no $v \in \mathbb{R}^N$ such that for each $x \in \psi(v)$, there exist some $i \in N$ and $v_i' \in \mathbb{R}$ for which

$$\min_{x' \in \psi(v_i', v_{-i})} u_i(x_i') > u_i(x_i).$$

Let us call this axiom *point-wise-ESP*. In the definition of *point-wise-ESP*, manipulation is considered for each point in $\psi(v)$ and manipulators can be different at every point. Thus, the idea of *point-wise-ESP* is based on the manipulation of an allocation in a chosen set rather than the set itself, unlike our ESP axioms.

Tadenuma and Thomson (1995, Theorem 1) show that, when there is only one indivisible good, no $\psi \subseteq F$ satisfies *point-wise-ESP*. By a proof similar to the proof of Proposition 5, one can easily show that no $\psi \subseteq PB$ satisfies *point-wise-ESP*. This generalizes Tadenuma and Thomson's result by allowing several indivisible objects to exist and extending F to PB.

5.2. Stochastic choice

In voting environments, Gibbard (1977) considers single-valued solutions that select a probability distribution over the set of alternatives.[17] This stochastic approach is quite different from our approach that selects a set of deterministic alternatives (allocations). However, when we compare sets taking into account probability distributions, there is no big difference between the two. We here restrict our attention to uniform distributions.[18]

Let

$$\mathcal{F} \equiv \{ f : \exists A \in \mathcal{X}, f \text{ is the uniform distribution over } A \}$$

be the set of uniform distributions whose supports belong to $\mathcal{X}$. Given $f \in \mathcal{F}$, let

$$\text{supp } f \equiv \{ x \in X : f(x) > 0 \}$$

be the support of f. A *stochastic solution* is a function h from $\mathbb{R}^N$ to $\mathcal{F}$. We assume that each agent evaluates every $f \in \mathcal{F}$ by expected utility. A stochastic solution is *stochastically strategy-proof* if there exist no $v \in \mathbb{R}^N$, $i \in N$, and v_i' such that, when $f \equiv h(v)$ and $f' \equiv h(v_i', v_{-i})$,

$$\int_{\text{supp } f'} u_i(x_i) f'(x) \mathrm{d}x > \int_{\text{supp } f} u_i(x_i) f(x) \mathrm{d}x.$$

Given a stochastic solution h, define the solution $\psi^h : \mathbb{R}^N \rightarrow X$ as $\psi^h(v) \equiv \text{supp} h(v)$ for each $v \in \mathbb{R}^N$. Then it is easy to see that ψ^h satisfies *expected-utility-ESP* if and only if h satisfies *stochastic strategy-proofness*. Therefore, Proposition 5 implies that there exists no stochastic solution h that satisfies *stochastic strategy-proofness*, "ex-post efficiency" ($\forall v \in \mathbb{R}^N$, $\text{supp} h(v) \subseteq P(v)$), and the "ex-post identical preferences lower bound" ($\forall v \in \mathbb{R}^N$, $\text{supp} h(v) \subseteq B(v)$).[19]

5.3. Implementability and ESP axioms

Nehring (2000) shows that Maskin's (1999) monotonicity condition for Nash implementation implies *separation-ESP* in voting environments.[20] It is unknown if the same implication holds in our environment. As far as we know, all solutions satisfying the monotonicity condition, such as F or PB, also satisfy *separation-ESP*.[21]

Jackson (1992) shows that *max-ESP* is necessary for implementation in undominated strategies using "bounded mechanism" in general environments. We observed that F and PB satisfy *max-*

[17] We refer to Barberà, Dutta, and Sen (2001) and Dutta, Peters, and Sen (2006) for recent developments on this topic.
[18] This approach is taken by Feldman (1980).
[19] These "ex-post" axioms ensure that the finally selected allocation is always normatively appealing by restricting the support of a chosen distribution. We do not know what happens if we consider "ex-post" versions of the axioms here.
[20] This result is originally shown by Muller and Satterthwaite (1977) under single-valuedness. Nehring generalizes the result for multi-valued solutions.
[21] Ohseto (2004) obtains impossibility results on Nash implementation of subsolutions of E.

ESP. However, it is open if these solutions are implementable in undominated strategies using "bounded mechanism".

5.4. Summary of results

The next table summarizes which main solutions satisfy which ESP axioms:

Table: satisfaction of ESP axioms

ESP\solutions	PE^{α}, PE^{β}	*PE*	*F*	*PB*
Separation	−	+	+	+
Min	−	−	−	−
Max	−	−	+	+
Weak no-addition	−	−	+	+
No-addition	−	−	+	−
No-deletion	−	−	−	−
Expected-utility	−	−	−	−

6. Conclusion

We studied strategic manipulations of (multi-valued) solutions in economies with homogeneous indivisible objects and monetary transfers. By examining the satisfaction of ESP axioms, we investigated how strategically manipulable solutions are and which types of manipulations occur. For example, we showed that the envy-free solution is robust to the manipulation that adds a better allocation to the set of chosen allocations, but it is vulnerable to the manipulation that deletes a worse allocation from the set of chosen allocations. Providing both positive and negative results is important here, since understanding what is behind a positive result passes through what cannot be done. This also enables us to compare the degree of manipulation across solutions. For instance, given our results, it is fair to say that the egalitarian-equivalent solution is more manipulable than the envy-free solution.

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